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CREEP IN STRUCTURES WITH RANDOM MATERIAL PROPERTIES

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Abstract—A material with randomly varying properties in creep is considered. A discrete hyperstatic structure, with one redundant member, and a thick-walled cylinder with internal pressure under stationary creep are studied. The expected values and the variances of the stresses and deformation rates are determined with a perturbation technique.

1. INTRODUCTION

During creep testing at constant stress levels the scatter in creep deformation rate and creep rupture time, is large for most materials. Experimental observations of local variations in deformation rate are presented by Chang and Grant[1]. They observed creep rates that varied along the specimen. Walles[2] gave the scatter between different test specimens a thorough statistical treatment. Observations from a number of creep tests[3], show that the shape of the distribution function of the measured strain rate is independent of the stress level. Walles has shown that the scatter is log-normal distributed, i.e. the logarithm of the strain rate is normal distributed. From these observations follow, assuming Norton's creep law $\dot{\epsilon} = B\sigma^n$ is valid, that the scatter originates from variations in B and that n can be treated as a constant.

Björkenstam[4, 5] studied the scatter due to load variations. He considered a load consisting of a constant part and a superposed randomly varying small part. The expected values and the variances of the stresses and the deformation rates for some structural elements were determined. The material was assumed to obey the Hooke–Norton constitutive equation $\dot{\epsilon} = \dot{\sigma}/E + B\sigma^n$.

Broberg and Westlund[6] assumed that the scatter was due to a random spatial variation of material properties in creep. The expected value and variance of the deformation rate of an ordinary test specimen under steady state creep were determined. A volume effect was shown to exist, viz. the variance of the deformation rate was shown to decrease with increased specimen size. The material was assumed to obey a modified Hooke–Norton constitutive equation as suggested by Broberg[3]. The same formulation was shown to be valid for a specimen in a random temperature field, for a material with the temperature dependence suggested by Dorn[7].

In the present paper the influence of random material properties on the deformation rate of a simple hyperstatic structure and a thick-walled cylinder under internal pressure is analysed. The scatter is assumed to be small, whereby the stresses may be solved by a perturbation technique used by Björkenstam[4, 5]. The expected value and variance of the deformation rate are determined.

2. MODEL MATERIAL

2.1 *Uniaxial behaviour*

A material with random properties under creep is considered. The total strain is expressed as an elastic part and a creep part

$$\epsilon(x) = \sigma(x)/E + \epsilon_c(x) \tag{1}$$

where x is the axial coordinate. The creep strain rate is given by the modified Norton creep equation

$$\dot{\epsilon}_c(x) = \dot{\epsilon}_0[1 + \alpha H(x)]\left(\frac{\sigma}{\sigma_n}\right)^n. \tag{2}$$

Here σ_n is a constant introduced for dimensional purposes and $\dot{\epsilon}_0$ and n are material constants. Moreover $\alpha H(x)$ is a normal distributed ergodic stochastic process, with the property $|\alpha H(x)| \ll 1$. A Markov type process is assumed.

Steady state creep conditions are considered. The total strain rate is then given by eqn (2). The statistical properties, i.e. the expected value, variance and autocorrelation, of the strain rate can be deduced from the known statistical properties of αH (see Broberg and Westlund[6]),

$$E[\alpha H] = \alpha \tag{3}$$

$$\text{Var}\,[\alpha H] = 2\alpha + O(\alpha^2) \tag{4}$$

$$R_\alpha H(x_0) = 2\alpha\, \mathrm{e}^{-\beta|x_0|} + O(\alpha^2). \tag{5}$$

Here x_0 is the distance between the two points considered and β is a material constant.

The statistical properties of the mean strain rate

$$\dot{\epsilon}_L = \dot{\epsilon}_0(1 + \alpha H_L)\left(\frac{\sigma}{\sigma_n}\right)^n \tag{6}$$

where

$$H_L = \frac{1}{L}\int_0^L H(x)\,\mathrm{d}x \tag{7}$$

are deduced from the statistical properties of αH_L. From eqns (3)–(5) follow

$$E[\alpha H_L] = \frac{1}{L}\int_0^L E[\alpha H(x)]\,\mathrm{d}x = \alpha \tag{8}$$

$$\text{Var}\,[\alpha H_L] = 2\alpha V(\beta L) + O(\alpha^2) \tag{9}$$

where

$$V(\beta L) = \frac{2}{(\beta L)^2}(\beta L - 1 + \mathrm{e}^{-\beta L}) \tag{10}$$

2.2 *Multiaxial behaviour*

Due to the manufacturing process, the material properties in creep will be inhomogeneous. Often a variation of material properties in one principal direction, denoted by x_1, will dominate. For cylindrical tubes this may be the radial direction.

The principal creep strain rates are postulated as

$$\dot{\epsilon}_{ci}(x_1, x_2, x_3) = \dot{\epsilon}_0\frac{3}{2}\left[\frac{\sigma_e(x_1, x_2, x_3)}{\sigma_n}\right]^n \frac{s_i(x_1, x_2, x_3)}{\sigma_e(x_1, x_2, x_3)}[1 + \alpha H(x_1)] \quad (i = 1, 2, 3) \tag{11}$$

where s_i and σ_e are the stress deviator and the Mises effective stress.

Steady state creep conditions are considered. The total strain rate is then given by eqn (11). The statistical properties of $\alpha H(x_1)$ are the same as in eqns (3)–(5).

3. HYPERSTATIC BAR SYSTEMS

3.1 *Basic equations*

A general hyperstatic structure with one redundant member, e.g. as in Fig. 1, is considered.

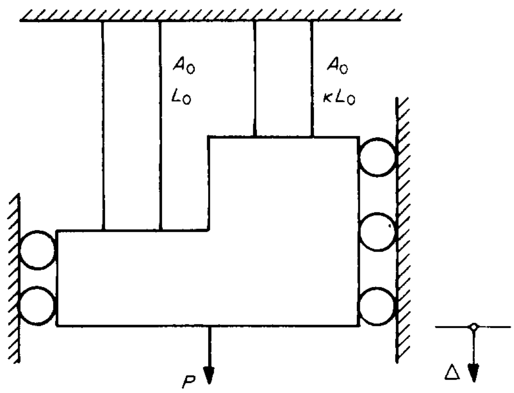

Fig. 1. Simple hyperstatic bar system.

The equilibrium condition can be written

$$\xi_1\sigma_1 + \xi_2\sigma_2 = P/A_0. \tag{12}$$

Here P is a load, A_0 is a reference area and ξ_1 and ξ_2 are geometrical constants depending on the structural shape.

The equation of compatibility can be written

$$\eta_1\epsilon_{L1} = \eta_2\epsilon_{L2} = \Delta/L_0. \tag{13}$$

Here Δ is a deflection under the load P, L_0 is a reference length and η_1 and η_2 are geometrical constants depending on the structural shape.

Insertion of the constitutive eqn (6) yields

$$\dot{\Delta}/L_0 = \eta_1\dot{\epsilon}_0(1+\alpha H_{L1})(\sigma_1/\sigma_n)^n = \eta_2\dot{\epsilon}_0(1+\alpha H_{L2})(\sigma_2/\sigma_n)^n. \tag{14}$$

Introduction of the non-dimensional quantities

$$s_i = \sigma_i/\sigma_n \quad (i=1,2), \qquad F = P/A_0\sigma_n, \qquad \dot{\lambda} = \dot{\Delta}/L_0\dot{\epsilon}_0 \tag{15}$$

transforms eqns (12) and (14) to

$$\xi_1 s_1 + \xi_2 s_2 = F \tag{16}$$

$$\dot{\lambda} = \eta_1(1+\alpha H_{L1})s_1{}^n = \eta_2(1+\alpha H_{L2})s_2{}^n. \tag{17}$$

3.2 *Stresses and deformation rate*

The stresses are written in the form of a series

$$s_i = s_{i0} + \alpha s_{i1} + \alpha^2 s_{i2} + O(\alpha^3) \quad (i=1,2). \tag{18}$$

Insertion in eqns (16) and (17), and identification of terms of equal order in α, yield

$$\xi_1 s_{10} + \xi_2 s_{20} = F \tag{19}$$

$$\xi_1 s_{11} + \xi_2 s_{21} = 0 \tag{20}$$

$$\xi_1 s_{12} + \xi_2 s_{22} = 0 \tag{21}$$

$$\eta_1 s_{10}^n = \eta_2 s_{20}^n \tag{22}$$

$$H_{L1} + n\frac{s_{11}}{s_{10}} = H_{L2} + n\frac{s_{21}}{s_{20}} \tag{23}$$

$$\frac{s_{11}}{s_{10}}H_{L1}+\frac{n-1}{2}\left(\frac{s_{11}}{s_{10}}\right)^2+\frac{s_{12}}{s_{10}}=\frac{s_{21}}{s_{20}}H_{L2}+\frac{n-1}{2}\left(\frac{s_{21}}{s_{20}}\right)^2+\frac{s_{22}}{s_{20}}. \tag{24}$$

The deformation rate is

$$\dot{\lambda}=\eta_1 s_{10}^n\left[1+n\frac{\alpha s_{11}}{s_{10}}+\alpha H_{L1}+n\frac{\alpha s_{11}\alpha H_{L1}}{s_{10}}+n\frac{\alpha^2 s_{12}}{s_{10}}+\frac{n(n-1)}{2}\left(\frac{\alpha s_{11}}{s_{10}}\right)^2\right]+O(\alpha^3). \tag{25}$$

The stresses are solved from eqns (19)–(24) as

$$s_{i0}=\frac{F\eta_j^{1/n}}{\xi_1\eta_2^{1/n}+\xi_2\eta_1^{1/n}}\quad (i,j=1,2;\ i\neq j) \tag{26}$$

$$s_{i1}=\frac{(H_{Lj}-H_{Li})s_{10}s_{20}\xi_j}{n(\xi_1 s_{10}+\xi_2 s_{20})}\quad (i,j=1,2;\ i\neq j) \tag{27}$$

$$s_{i2}=\frac{s_{10}s_{20}\xi_j}{2n^2(\xi_1 s_{10}+\xi_2 s_{20})^2}\{[(n-1)\xi_i s_{i0}+(n+1)\xi_j s_{j0}](H_{Li})^2$$
$$+2(\xi_i s_{i0}-\xi_j s_{j0})H_{Li}H_{Lj}-[(n+1)\xi_i s_{i0}+(n-1)\xi_j s_{j0}](H_{Lj})^2\}\quad (i,j=1,2;\ i\neq j). \tag{28}$$

The stresses of eqn (26) are recognized as the solution for the scatter-free material.

3.3 *The statistical properties of stresses and deformation rate*

The expected values and variances of the stresses and deformation rate, deduced from eqns (8) and (9) and (25)–(28), are

$$E[s_{i0}]=s_{i0}\qquad (i=1,2) \tag{29}$$

$$\mathrm{Var}\,[s_{i0}]=0\qquad (i=1,2) \tag{30}$$

$$E[\alpha s_{i1}]=O(\alpha^2)\ \ (i=1,2) \tag{31}$$

$$\mathrm{Var}\,[\alpha s_{i1}]=\frac{2}{n^2}\left(\frac{s_{10}s_{20}\xi_j}{\xi_1 s_{10}+\xi_2 s_{20}}\right)^2\alpha(V_j+V_i)+O(\alpha^2)\quad (i,j=1,2;i\neq j) \tag{32}$$

where

$$V_i=V(\beta L_i) \tag{33}$$

$$E[\alpha^2 s_{i2}]=\frac{s_{10}s_{20}\xi_j\alpha}{n^2(\xi_1 s_{10}+\xi_2 s_{20})^2}\{[(n-1)\xi_i s_{i0}+(n+1)\xi_j s_{j0}]V_i$$
$$-[(n+1)\xi_i s_{i0}+(n-1)\xi_j s_{j0}]V_j\}+O(\alpha^2)\quad (i,j=1,2;\ i\neq j) \tag{34}$$

$$\mathrm{Var}\,[\alpha^2 s_{i2}]=O(\alpha^2)\quad (i=1,2) \tag{35}$$

and

$$E[\dot{\lambda}]=\eta_1 s_{10}^n\left\{1+\alpha\left[1-\frac{n+1}{n}\frac{\xi_1\xi_2 s_{10}s_{20}}{(\xi_1 s_{10}+\xi_2 s_{20})^2}(V_1+V_2)\right]\right\}+O(\alpha^2) \tag{36}$$

$$\mathrm{Var}\,[\dot{\lambda}]=2\eta_1^2 s_{10}^{2n}\alpha\left[\frac{\xi_1^2 s_{10}^2+2\xi_1\xi_2 s_{10}s_{20}}{(\xi_1 s_{10}+\xi_2 s_{20})^2}V_1+\frac{\xi_2^2 s_{20}^2}{(\xi_1 s_{10}+\xi_2 s_{20})^2}V_2\right]+O(\alpha^2). \tag{37}$$

Insertion of eqn (26) yields for the deformation rate

$$E[\dot{\lambda}]=\frac{F^n\eta_1\eta_2}{(\xi_1\eta_2^{1/n}+\xi_2\eta_1^{1/n})^n}\left\{1+\alpha\left[1-\frac{n+1}{n}\frac{\xi_1\xi_2\eta_1^{1/n}\eta_2^{1/n}}{(\xi_1\eta_2^{1/n}+\xi_2\eta_1^{1/n})^2}(V_1+V_2)\right]\right\}+O(\alpha^2) \tag{38}$$

$$\mathrm{Var}\,[\dot{\lambda}]=\frac{2F^{2n}\eta_1^2\eta_2^2}{(\xi_1\eta_2^{1/n}+\xi_2\eta_1^{1/n})^{2n}}\alpha\left[\frac{\xi_1^2\eta_2^{2/n}+2\xi_1\xi_2\eta_1^{1/n}\eta_2^{1/n}}{(\xi_1\eta_2^{1/n}+\xi_2\eta_1^{1/n})^2}V_1+\frac{\xi_2^2\eta_1^{2/n}}{(\xi_1\eta_2^{1/n}+\xi_2\eta_1^{1/n})^2}V_2\right]+O(\alpha^2). \tag{39}$$

The second and the third factor, of the second term inside the square bracket of eqn (38), represent the influence of the structural shape and size. The two terms in the square bracket of eqn (39) are products of structural shape and size factor.

3.4 *Application to a simple structure*

The structure in Fig. 1 is considered. The geometrical constants are

$$\xi_1 = \xi_2 = 1; \quad \eta_1 = 1; \quad \eta_2 = \kappa. \tag{40}$$

Equations (38) and (39) yield

$$E[\dot{\lambda}] = \frac{F^n \kappa}{(1+\kappa^{1/n})^n}\left\{1 + \alpha\left[1 - \frac{n+1}{n}\frac{\kappa^{1/n}}{(1+\kappa^{1/n})^2}(V_1 + V_2)\right]\right\} + O(\alpha^2) \tag{41}$$

$$\mathrm{Var}[\dot{\lambda}] = 2\frac{F^{2n}\kappa^2}{(1+\kappa^{1/n})^{2n}}\alpha\left\{\frac{\kappa^{2/n} + 2\kappa^{1/n}}{(1+\kappa^{1/n})^2}V_1 + \left[1 - \frac{\kappa^{2/n} + 2\kappa^{1/n}}{(1+\kappa^{1/n})^2}\right]V_2\right\} + O(\alpha^2). \tag{42}$$

The expected value and variance are always smaller for the structure than for the simple specimen with load $F/(1+\kappa^{-1/n})$ and length L_2. This enables conservative design.

The structural shape and size factors can be read from Figs. 2 and 3.

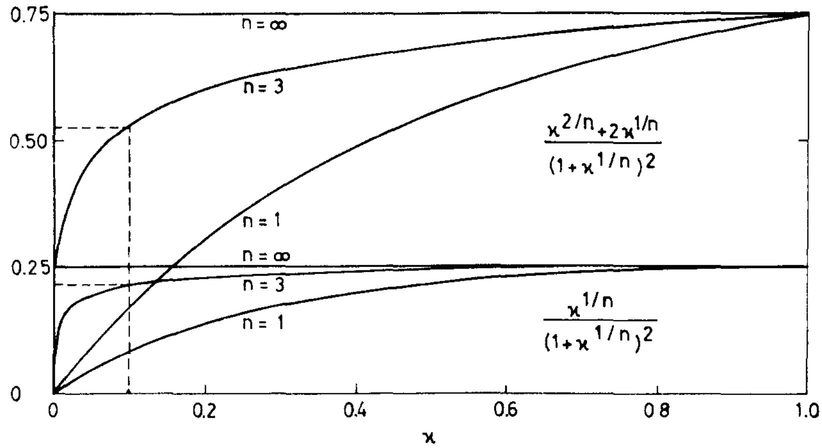

Fig. 2. Structural shape factors.

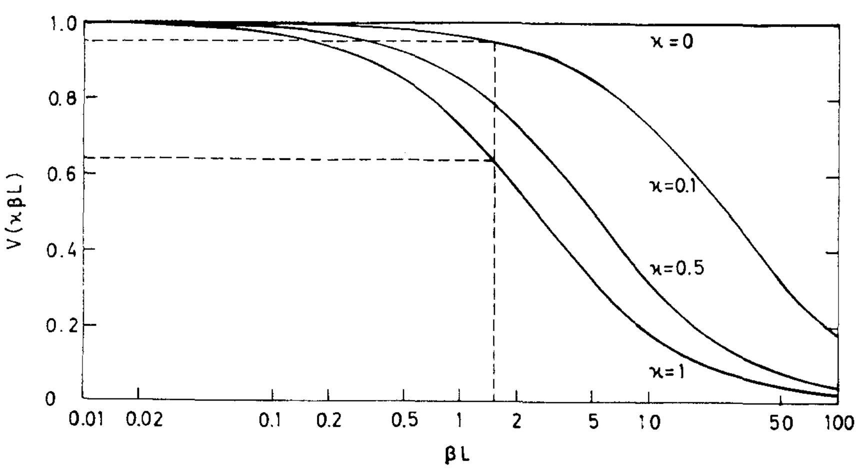

Fig. 3. Structural size factor.

When $\kappa \to 1$ the expected value tends to

$$E[\dot{\lambda}] = (F/2)^n \left\{ 1 + \alpha \left[1 - \frac{n+1}{n} \frac{1}{2} V_1 \right] \right\} + O[\alpha^2) \tag{43}$$

which is always smaller than the expected value of the single specimen with load $F/2$

$$E[\dot{\lambda}] = (F/2)^n (1 + \alpha) + O(\alpha^2). \tag{44}$$

This is due to the two bars being more constrained than the single specimen. They are forced to have the same total deformation rate, though the local deformation rates can differ between them. When L_1 is large enough eqn (43) converges towards eqn (44).

The variance tends to

$$\text{Var}\,[\dot{\lambda}] = 2(F/2)^{2n} \alpha V_1 + O(\alpha^2) \tag{45}$$

which is identical with the variance of the single specimen with load $F/2$.

It is seen from eqns (41) to (45) that the expected value and the variance, of the deformation rate, are always smaller for the structure then for the isostatic structural elements.

4. THICK-WALLED CYLINDER

4.1 *Basic equations*

A thick-walled cylinder is considered subjected to internal pressure p and zero axial strain. With standard assumptions (isotropy, isochoric deformation) and standard notation (radial coordinate r, circumferential coordinate ϕ) the equations of equilibrium and compatibility are

$$\sigma_\phi = \frac{\mathrm{d}}{\mathrm{d}r}(r\sigma_r) \tag{46}$$

$$\epsilon_r = \frac{\mathrm{d}}{\mathrm{d}r}(r\epsilon_\phi). \tag{47}$$

The constitutive equation (11) takes the forms

$$\dot{\epsilon}_r = \dot{\epsilon}_0 \frac{3}{2} \left(\frac{\sigma_e}{\sigma_n} \right)^n \frac{s_r}{\sigma_e} [1 + \alpha H(r)] \tag{48}$$

$$\dot{\epsilon}_\phi = \dot{\epsilon}_0 \frac{3}{2} \left(\frac{\sigma_e}{\sigma_n} \right)^n \frac{s_\phi}{\sigma_e} [1 + \alpha H(r)]. \tag{49}$$

The stress deviators and the Mises effective stress are

$$s_\phi = -s_r = \frac{1}{2}(\sigma_\phi - \sigma_r) \tag{50}$$

$$\sigma_e = \frac{\sqrt{3}}{2}(\sigma_\phi - \sigma_r). \tag{51}$$

The radial displacement rate is

$$\dot{u} = r\dot{\epsilon}_\phi = C/r. \tag{52}$$

Insertion of eqns (48)–(51) in eqn (47) yields

$$rn(1 + \alpha H)(\sigma'_\phi - \sigma'_r) + r\alpha H'(\sigma_\phi - \sigma_r) + 2(1 + \alpha H)(\sigma_\phi - \sigma_r) = 0 \tag{53}$$

where the prime denotes $\mathrm{d}/\mathrm{d}r$.

Elimination of σ_ϕ with the aid of eqn (46) gives a differential equation for the radial stress

$$r(1+\alpha H)\sigma_r'' + \frac{2+n}{n}(1+\alpha H)\sigma_r' + \frac{r}{n}\alpha H'\sigma_r' = 0 \qquad (54)$$

with the boundary conditions

$$\sigma_r(a) = -p; \quad \sigma_r(b) = 0. \qquad (55)$$

4.2 *Stresses and displacement rate*

The radial stress is written in the form of a series

$$\sigma_r = \sigma_{r0} + \alpha\sigma_{r1} + \alpha^2\sigma_{r2} + O(\alpha^3). \qquad (56)$$

Insertion in eqn (54), and identification of terms of equal order in α, yield three differential equations that can be solved successively

$$r\sigma_{r0}'' + \frac{2+n}{n}\sigma_{r0}' = 0 \qquad (57)$$

$$r\sigma_{r1}'' + \frac{2+n}{n}\sigma_{r1}' = -\frac{r}{n}H'\sigma_{r0}' \qquad (58)$$

$$r\sigma_{r2}'' + \frac{2+n}{n}\sigma_{r2}' = -\frac{r}{n}H'\sigma_{r1}' + \frac{r}{n}H'H\sigma_{r0}'. \qquad (59)$$

The solution of eqn (57) is

$$\sigma_{r0} = A_0 - B_0 r^{-2/n}. \qquad (60)$$

The boundary conditions

$$\sigma_{r0}(a) = -p; \quad \sigma_{r0}(b) = 0 \qquad (61)$$

give

$$B_0 = p/(a^{-2/n} - b^{-2/n}) \qquad (62)$$

$$A_0 = b^{-2/n}B_0. \qquad (63)$$

Insertion of $\sigma_r'{}_0$ in eqn (58) gives

$$r^{-2/n}\frac{\mathrm{d}}{\mathrm{d}r}(r^{1+2/n}\sigma_{r1}') = -\frac{2B_0}{n^2}H'r^{-2/n} \qquad (64)$$

with the solution

$$\sigma_{r1} = A_1 - B_1 r^{-2/n} - \frac{2B_0}{n^2}\int_a^r H\xi^{-1-2/n}\,\mathrm{d}\xi. \qquad (65)$$

The boundary conditions

$$\sigma_{r1}(a) = \sigma_{r1}(b) = 0 \qquad (66)$$

yield

$$B_1 = \frac{2B_0}{n^2(a^{-2/n} - b^{-2/n})}\int_a^b H\xi^{-1-2/n}\,\mathrm{d}\xi \qquad (67)$$

$$A_1 = B_1 a^{-2/n}. \qquad (68)$$

Insertion of σ'_{r0} and σ'_{r1} in eqn (59) gives

$$r^{-2/n}\frac{\mathrm{d}}{\mathrm{d}r}(\sigma'_{r2}r^{1+2/n}) = \frac{2B_0(n+1)}{n^3}H'Hr^{-2/n} - \frac{2B_1}{n^2}H'r^{-2/n} \tag{69}$$

with the solution

$$\sigma_{r2} = A_2 - B_2r^{-2/n} + \frac{B_0(n+1)}{n^3}\int_a^r H^2\xi^{-1-2/n}\,\mathrm{d}\xi - \frac{2\alpha B_1}{n^2}\int_a^r H\xi^{-1-2/n}\,\mathrm{d}\xi. \tag{70}$$

The boundary conditions

$$\sigma_{r2}(a) = \sigma_{r2}(b) = 0 \tag{71}$$

yield

$$B_2 = -\frac{B_0(n+1)}{n^3(a^{-2/n} - b^{-2/n})}\int_a^b H^2\xi^{-1-2/n}\,\mathrm{d}\xi + \frac{2B_1}{n^2(a^{-2/n} - b^{-2/n})}\int_a^b H\xi^{-1-2/n}\,\mathrm{d}\xi \tag{72}$$

$$A_2 = B_2a^{-2/n}. \tag{73}$$

The stress of eqn (60) is recognized as the solution for the scatter-free material.

The radial stress is

$$\sigma_r = A_0 - B_0r^{-2/n} + \alpha A_1 - \alpha B_1r^{-2/n} - \frac{2}{n^2}B_0\int_a^r \alpha H\xi^{-1-2/n}\,\mathrm{d}\xi + \alpha^2A_2 - \alpha^2B_2r^{-2/n}$$
$$+\frac{n+1}{n^2}\int_a^r(\alpha H)^2\xi^{-1-2/n}\,\mathrm{d}\xi - \frac{2\alpha B_1}{n}\int_a^r \alpha H\xi^{-1-2/n}\,\mathrm{d}\xi + O(\alpha^3). \tag{74}$$

The circumferential stress can be solved from e.g. (46) as

$$\sigma_\phi - \sigma_r = r\sigma'_r = \frac{2B_0}{n}r^{-2/n}\left[1 + \frac{\alpha B_1}{B_0} - \frac{1}{n}\alpha H + \frac{\alpha^2B_2}{B_0} + \frac{(n+1)}{2n^2}(\alpha H)^2 - \frac{\alpha B_1}{nB_0}\alpha H\right] + O(\alpha^3). \tag{75}$$

Equations (49)–(52) and eqn (75) yield for the displacement rate

$$\dot{u} = \frac{C_0}{r}(1+\alpha H)\left[1 + \frac{\alpha B_1}{B_0} - \frac{\alpha H}{n} + \frac{\alpha^2B_2}{B_0} + \frac{n+1}{2n^2}(\alpha H)^2 - \frac{\alpha B_1}{nB_0}\alpha H\right]^n + O(\alpha^3) \tag{76}$$

where

$$C_0 = \frac{\sqrt{3}}{2}\dot{\epsilon}_0\left(\frac{\sqrt{3}B_0}{n\sigma_n}\right)^n. \tag{77}$$

A series development of the square bracket yields

$$\dot{u} = \frac{C_0}{r}\left[1 + n\frac{\alpha B_1}{B_0} + n\frac{\alpha^2B_2}{B_0} + \frac{n(n-1)}{2}\left(\frac{\alpha B_1}{B_0}\right)^2\right] + O(\alpha^3). \tag{78}$$

Insertion of the constants and introduction of

$$I_k = \int_a^b (\alpha H)^k\xi^{-1-2/n}\,\mathrm{d}\xi \quad (k = 0, 1, 2) \tag{79}$$

gives

$$\dot{u} = \frac{C_0}{r}\left[1 + \frac{I_1}{I_0} - \frac{n+1}{2n}\frac{I_2}{I_0} + \frac{n+1}{2n}\left(\frac{I_1}{I_0}\right)^2\right] + O(\alpha^3). \tag{80}$$

4.3 *The statistical properties of the displacement rate*

The statistical properties of $\dot{u}$ can be deduced with the aid of

$$E[I_1] = \int_a^b E[\alpha H]\xi^{-1-2/n}\,\mathrm{d}\xi = \alpha I_0 \quad (81)$$

$$E[I_2] = \int_a^b E[(\alpha H)^2]\xi^{-1-2/n}\,\mathrm{d}\xi = 2\alpha I_0 + O(\alpha^2) \quad (82)$$

$$E[I_1^2] = \int_a^b \int_a^b E[\alpha H(\xi_1)\alpha H(\xi_2)]\xi_1^{-1-2/n}\xi_2^{-1-2/n}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2. \quad (83)$$

By definition

$$E[\alpha H(r_1)\alpha H(r_2)] = R_{\alpha H}(r_1 - r_2). \quad (84)$$

Insertion of the autocorrelation from eqn (5) yields

$$E[I_1^2] = 2\alpha I_0^2 V[\beta(b-a)] + O(\alpha^2) \quad (85)$$

where

$$V[\beta(b-a)] = \frac{4\int_{1/(\kappa-1)}^{\kappa/(\kappa-1)}\int_{1/(\kappa-1)}^{\kappa/(\kappa-1)} \mathrm{e}^{-\beta(b-a)|\zeta_1-\zeta_2|}(\zeta_1\zeta_2)^{-1-2/n}\,\mathrm{d}\zeta_1\,\mathrm{d}\zeta_2}{n^2(\kappa-1)^{4/n}(1-\kappa^{-2/n})^2} \quad (86)$$

and

$$\kappa = b/a. \quad (87)$$

The variance of I_1, I_2 and I_1^2 is deduced from eqns (81)–(85)

$$\mathrm{Var}\,[I_1] = 2\alpha I_0^2 V + O(\alpha^2) \quad (88)$$

$$\mathrm{Var}\,[I_2] = O(\alpha^2) \quad (89)$$

$$\mathrm{Var}\,[I_1^2] = O(\alpha^2). \quad (90)$$

Thus

$$E[\dot{u}] = \frac{C_0}{r}\left\{1 + \alpha\left[1 - \frac{n+1}{n}(1-V)\right]\right\} + O(\alpha^2) \quad (91)$$

$$\mathrm{Var}\,[\dot{u}] = \left(\frac{C_0}{r}\right)^2 2\alpha V + O(\alpha^2). \quad (92)$$

The expected value and variance are always smaller for the structure than for the simple specimen with load $(C_0/r)^{1/n}$ and length $\kappa^{-1/n}(b-a)$. This enables conservative design.

The structural geometry factor V can be read from Fig. 4. The close similarity between the bottom curves of Figs. 3 and 4 should be noted. This indicates that similar results may be expected for other simple redundant structures.

5. DISCUSSION

Inhomogeneous creep properties have been considered in order to describe the observed scatter in structural creep deformation rates. Norton's creep law has been formulated as a stochastic process. The inhomogeneity has been described as a random variation of material properties in one principal direction only. This simplification may often be justified by the manufacturing process of the structural elements. The inhomogeneity also may appear due to a random temperature field.

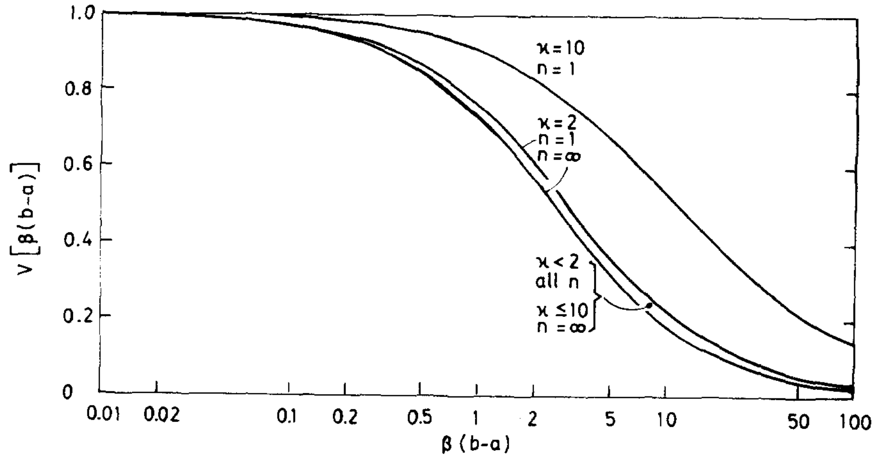

Fig. 4. Structural geometry factor.

The stresses and deformation rates of two simple hyperstatic structures, viz. a two-bar system and a thick-walled cylinder, under steady state creep have been determined. The influence of the local variations of the material properties on the expected values and variances of the deformation rates have been analysed. A structural shape and size effect have been shown to exist, viz. the dependence of the scatter in material properties is decreased when the structural redundancy and size is increased. The influence on the expected values is always small, but the influence on the variances may be large. Estimations of the structural behaviour from the simple specimen has been shown to be conservative as long as the specimen length and load are correctly chosen.

It is possible, with the same technique, to consider multi-layered cylinders. A straightforward extension to creep with random material properties in all principal directions may be done.

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